Calculation of Transonic Internal Flows Using an Efficient High Resolution Upwind Scheme

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Objective:

- Develop an E-CUSP upwind scheme with high accuracy and efficiency

Background:

- Aircraft and engine design need CFD solver with high efficiency and accuracy
- Roe scheme popular for transonic flows with high resolution for discontinuities
- More efficient schemes with scalar dissipation:
  
  H-CUSP schemes: Liou’s AUSM family scheme, Edwards’ LDFSS schemes, Van Leer-Hänel scheme, Jameson’s H-CUSP schemes


  Flux Vector schemes: Steger-Warming scheme, Van Leer scheme
• H-CUSP schemes (e.g. AUSM family schemes) have high accuracy, but not fully consistent with characteristics

• E-CUSP scheme is consistent with characteristics. Previous E-CUSP scheme is not smooth, or not able to capture the contact surfaces.

• This paper is to develop an E-CUSP scheme which is efficient, accurate and robust.
Governing Equations

Quasi-1D Euler equations

$$\partial_t U + \partial_x E - H = 0$$  \hspace{1cm} (1)

where \( U = SQ \), \( Q = \begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix} \), \( E = SF \),

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho e + p)u \end{pmatrix}, \quad H = \frac{dS}{dx} \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix}$$  \hspace{1cm} (2)

Explicit finite volume method

$$\Delta Q_i^{n+1} = \Delta t \left[ -C \left( E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}} \right) + \frac{H_i}{S_i} \right]^n$$  \hspace{1cm} (3)
Characteristics

Jacobian matrix

\[
\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{Q}} = \mathbf{T} \Lambda \mathbf{T}^{-1}
\]  \hspace{1cm} (4)

where \( \mathbf{T} = \begin{pmatrix} 1 & 1 & 1 \\ u - a & u & u + a \\ H - ua & \frac{1}{2}u^2 & H + ua \end{pmatrix} \)

and

\[
\mathbf{A} = \begin{pmatrix} u - a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + a \end{pmatrix}
\]  \hspace{1cm} (5)
Flux Splitting

\[ F = T \Lambda T^{-1}Q \]  \hspace{1cm} (6)

\[ F = T \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} T^{-1}Q + T \begin{pmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} T^{-1}Q = F^c + F^p \]  \hspace{1cm} (7)

where

\[ F^c = u \begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix}, F^p = \begin{pmatrix} 0 \\ p \\ p u \end{pmatrix} \]  \hspace{1cm} (8)

\( F^c \) has eigenvalues \((u, u, u)\), convective term, upwind

\( F^p \) has eigenvalues \((-a, 0, a)\), acoustic wave (pressure) term, upwind and downwind.

This splitting naturally leads to E-CUSP.
\textbf{H-CUSP}

\[ \mathbf{F} = \mathbf{F}^{\nu c} + \mathbf{F}^{\nu p} \]  \hspace{1cm} (9)

\[ \mathbf{F}^{\nu c} = u \begin{pmatrix} \rho \\ \rho u \\ \rho H \end{pmatrix}, \quad \mathbf{F}^{\nu p} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \]  \hspace{1cm} (10)

where \( H \) is the total enthalpy

\[ H = \frac{\rho c + p}{\rho} \]  \hspace{1cm} (11)

\( \mathbf{F}^{\nu c} \) has eigenvalues \((u, u, \gamma u)\), upwind

\( \mathbf{F}^{\nu p} \) has eigenvalues \((0, 0, -(\gamma - 1)u)\), downwind
The New E-CUSP Scheme

For $|u| \leq a$,

$$
\mathbf{F}_{1/2} = \frac{1}{2}[(\rho u)_{1/2}(\mathbf{q}^c_L + \mathbf{q}^c_R) - |\rho u|_{1/2}(\mathbf{q}^c_R - \mathbf{q}^c_L)]
+ \left( \begin{array}{c}
0 \\
\mathcal{P}^+ p \\
\frac{1}{2} p (u + a_1) 
\end{array} \right)_L
+ \left( \begin{array}{c}
0 \\
\mathcal{P}^- p \\
\frac{1}{2} p (u - a_1) 
\end{array} \right)_R
$$

(12)

For $u > a$, $\mathbf{F}_{1/2} = \mathbf{F}_L$; For $u < -a$, $\mathbf{F}_{1/2} = \mathbf{F}_R$

Interface mass flux is introduced based on Wada-Liou AUSMD scheme:

$$(\rho u)_{1/2} = (\rho_L u^+_L + \rho_R u^-_R)$$

(13)

$$u^+_L = a_1 \left\{ \frac{M_L + |M_L|}{2} + \alpha_L \left[ \frac{1}{4} (M_L + 1)^2 - \frac{M_L + |M_L|}{2} \right] \right\}$$

(14)

$$u^-_R = a_1 \left\{ \frac{M_R - |M_R|}{2} + \alpha_R \left[ -\frac{1}{4} (M_R - 1)^2 - \frac{M_R - |M_R|}{2} \right] \right\}$$

(15)
The New E-CUSP Scheme, continued

Interface speed of sound

\[ a_{1/2} = \frac{1}{2}(a_L + a_R) \]  \hspace{1cm} (16)

\[ M_L = \frac{u_L}{a_{1/2}}, \quad M_R = \frac{u_R}{a_{1/2}} \]  \hspace{1cm} (17)

\[ \alpha_L = \frac{2(p/\rho)_L}{(p/\rho)_L + (p/\rho)_R}, \quad \alpha_R = \frac{2(p/\rho)_R}{(p/\rho)_L + (p/\rho)_R} \]  \hspace{1cm} (18)

Pressure splitting in momentum eq.

\[ \mathcal{P}^\pm = \frac{1}{4}(M \pm 1)^2(2 \mp M) \pm \alpha M(M^2 - 1)^2, \quad \alpha = \frac{3}{16} \]  \hspace{1cm} (19)
Numerical Dissipation

At stagnation \( u = 0 \), the dissipation of the new scheme:

\[
D = -\frac{a_1^2}{2} \begin{pmatrix} 0 \\ 0 \\ \delta p \end{pmatrix} \tag{20}
\]

where

\[
\delta p = p_R - p_L \tag{21}
\]

The dissipation of the Roe scheme:

\[
D_{Roe} = -\frac{\tilde{a}_1^2}{2(\gamma - 1)} \begin{pmatrix} (\gamma - 1)/\tilde{a}_1^2 \delta p \\ 0 \\ \delta p \end{pmatrix} \tag{22}
\]

The dissipation of the new scheme is not greater than that of the Roe scheme.
The Sod Shock Tube Problem

Figure 1: Temperature, Zha E-CUSP scheme
The Sod Shock Tube Problem

Figure 2: Temperature, Roe scheme
The Sod Shock Tube Problem

Figure 3: Temperature, Van Leer scheme
The Sod Shock Tube Problem

Figure 4: Temperature, Van Leer-Hänel scheme
The Sod Shock Tube Problem

Figure 5: Temperature, Liou AUSM$^+$ scheme
The Sod Shock Tube Problem

Table 1: **Maximum CFL Numbers for Sod 1D Shock Tube**

<table>
<thead>
<tr>
<th>Scheme</th>
<th>CFL Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>The new scheme (Zha CUSP)</td>
<td>1.00</td>
</tr>
<tr>
<td>Van Leer-Hänel</td>
<td>1.00</td>
</tr>
<tr>
<td>Van Leer</td>
<td>0.96</td>
</tr>
<tr>
<td>Roe</td>
<td>0.95</td>
</tr>
<tr>
<td>Liou $AUSM^+$</td>
<td>0.275</td>
</tr>
</tbody>
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Slowly Moving Contact Surface

Figure 6: Density and velocity, E-CUSP, Roe, AUSM\(^+\) scheme
Figure 7: Density and velocity, Van Leer, Van Leer-Hanel scheme
Quasi-1D Nozzle, Mach number
Laminar Flat Plate, $M=2.0$, Velocity Profile
Laminar Flat Plate, $M=2.0$, Temperature Profile
NASA Transonic Nozzle, Mach Number Contours, New E-CUSP Scheme
NASA Transonic Nozzle, Wall Mach Number Distribution

![Graph showing the entropic Mach number distribution for different schemes and experimental data. The graph includes data for Zha CUSP 175x50, Zha CUSP 175x80, Roe Scheme 175x50, and Roe Scheme 175x80, compared to experimental data. The x-axis represents X/L, and the y-axis represents the isentropic Mach number.]
Inlet Diffuser, Mach Number Contours, $\frac{p_{out}}{p_t} = .83$
Inlet Diffuser, Surface Pressure Distribution, \( \frac{p_{out}}{p_t} = .83 \)
Inlet Diffuser, Surface Pressure Distribution, $p_{out}/p_t = .72$
Inlet Diffuser, Mach Number Contours, $p_{out}/p_t = .72$

Zha CUSP Scheme

Roe Scheme

AUSM+ Scheme
Conclusions:

- The new E-CUSP scheme is efficient and has low numerical dissipation.
- Able to capture crisp shock profile and exact contact discontinuities
- For 1D Sod shock problem, $CFL_{max} = 1$, crispest shock profile
- For quasi-1D nozzle, no expansion shock generated at sonic point.
- For $M=2$ laminar flat plate, 1st order scheme obtains accurate velocity and temperature profiles
- For a transonic nozzle, oblique shock captured well
- For a transonic inlet-diffuser with shock wave/turbulent boundary layer interaction, the surface pressure agree well with experiment.