We present simulations of Mach 12, 3D, hypersonic turbulent flows of air around blunt bodies with applied magnetic fields of various strengths. We use the Baldwin-Lomax turbulence model. The air is heated by a shock that can change its thermodynamic properties. We treat the air as a real gas in thermodynamic equilibrium. Due to the high temperature the thermodynamic properties of the air are affected changes in specific heats due to rotational and vibrational excitation, by molecular dissociation, ionization, and other effects. We calculate the thermodynamic properties of the air using the curve fitting method implemented in the TGAS FORTRAN subroutines. The hot air is partially ionized, but the electrical conductivity of the air is low enough that we can use a low magnetic Reynolds number MHD model. We use a temperature dependent electrical conductivity with an onset temperature of 4,000 K. Our numerical method is based on the Low Diffusion E-CUSP (LDE) scheme developed by Zha\textsuperscript{8} et al with a 5th order WENO scheme.

I. Introduction

Hypersonic vehicles generate shocks that can heat the air sufficiently to partially ionize the air and create an electrically conducting plasma that can be studied using the equations of single fluid magnetohydrodynamics (MHD). Introducing strong applied magnetic and electric fields into the flow could have beneficial effects such as reducing heat damage, providing a sort of MHD parachute, and generating electric power or thrust in the vehicle. This paper is a continuation of the work we presented in Ref. [1]. Among the many authors that have done numerical simulations of steady hypersonic MHD flows we mention here Bityurin, Lineberry, Potebnia, et. al.,\textsuperscript{2} Bityurin and Bocharov\textsuperscript{3}, Bityurin, Zeigarnik and Kuranov\textsuperscript{4}, Fujino, Kondo and Ishikawa\textsuperscript{5}, MacCormack\textsuperscript{6}, and Hoffmann, Damevin, and Dietiker\textsuperscript{7}. The low diffusion E-CUSP (LDE) scheme with a fifth order WENO scheme recently developed by
Zha\textsuperscript{8} et al. is very good in resolving flow fields with shock discontinuities. The purpose of this paper is to incorporate the low magnetic Reynolds number MHD model and the thermodynamics of high temperature air to the above CFD algorithm so that it can be used to simulate hypersonic flows with MHD effects. In this paper we extend our results of Ref. [1] to Mach 12, and we include the Balswin-Lomax turbulence model.

II. Treatment of Real Gas Effects

The high temperature behind the hypersonic bow shock causes molecular dissociation, chemical reactions, ionization, and other effects that change the equation of state and the energy equation of the gas. A complete treatment of this requires the inclusion of reaction rate equations for each chemical species, the inclusion of rotational and vibrational energy equations, and other effects, possibly with the presence of two or even three distinct temperatures. Here we simply assume that there is enough time for the gas to reach local thermodynamic equilibrium, and we calculate the gas properties for air in chemical and thermodynamic equilibrium using the the curve fitting method implemented in the TGAS FORTRAN subroutines developed by Srinivasan, Tannehill, and Weilmuenster\textsuperscript{9}, and Srinivasan, Tannehill, and Weilmuenster\textsuperscript{10}. These subroutines give the pressure $p$ and temperature $T$ in terms of an equivalent ratio of specific heats $\tilde{\gamma}$ in the form

$$\tilde{\gamma} = \tilde{\gamma}(e_{\text{int}}, \rho), \quad p = \rho e_{\text{int}}(\tilde{\gamma} - 1), \quad T = T(p, \rho),$$

where $\rho$ and $e_{\text{int}}$ are the gas density and specific internal energy. In this abstract we use Sutherland’s law to calculate the viscosity $\mu$.

III. The MHD Model

Single-fluid magnetohydrodynamics (MHD) describes an electrically conducting but electrically neutral fluid of density $\rho$, velocity $\mathbf{u}$, pressure $p$, energy per unit mass $e$, viscosity tensor $\bar{\tau}$, and heat flux vector $\mathbf{q}$. The electrical quantities are the magnetic field $\mathbf{B}$, the electric field $\mathbf{E}$, the current density $\mathbf{J}$, and the electrical conductivity $\sigma_e$, which may be a scalar or a tensor, depending on the model. In principle $\sigma_e$ should be calculated from an air chemistry model, such as in Ref [6], but in this paper we take a variable electrical conductivity model based on a power-law with an onset temperature $T_{\text{onset}} = 4,000$ K, and $T_0 = T_{\text{ref}}$, where $T_{\text{ref}}$ is the stagnation temperature in the flow with no magnetic field. We take $\sigma_{e,\text{ref}} = 100$ S/m.

$$\sigma_e = \sigma_{e,\text{ref}} \Theta(T - T_{\text{onset}}) \left( \frac{T}{T_0} \right)^k.$$

Except for the onset temperature, this model was introduced in the analytical studies by Bush\textsuperscript{11,12}, and further studied by Poggie and Gaitonde.\textsuperscript{13} The magnetic Reynolds number is $Re_m = \mu_\infty \sigma_e U_\infty L_\infty$. In the cases treated above $Re_m$ is small enough that the magnetic field is taken to be the externally applied field, and the magnetic field produced by the plasma current is neglected.
IV. The Governing Equations of a Three-dimensional Low Reₘ Approximation

A generalized coordinate systems

\[ \xi(x, y, z), \eta(x, y, z), \zeta(x, y, z) \] with \( J_a = \frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} \),

where \( J_a \) is the Jacobian determinant of the transformation of variables is used to convert the sophisticated grid in real space into a rectangular grid in computational space. The low diffusion E-CUSP (LDE) scheme developed by Zha, Shen, and Wang is used as described before. In this paper the Reynolds number is low enough that turbulence effects are negligible and laminar flow applies. The governing equations are the laminar 3D Navier-Stokes equations as follows.

\[
\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial \xi} + \frac{\partial F}{\partial \eta} + \frac{\partial G}{\partial \zeta} = \frac{1}{Re} \left( \frac{\partial R}{\partial \xi} + \frac{\partial S}{\partial \eta} + \frac{\partial T}{\partial \zeta} \right) + S_{MHD}. \tag{4}
\]

To find a steady state solution an appropriate flow is taken as an initial condition. The flow is developed in time using the above equation. The flow is considered to reached a steady state when the maximum change in any cell of each component is less than some \( \epsilon \). Here, \( \epsilon \approx 10^{-4} \sim 10^{-5} \). The flux vector is

\[
Q = \frac{1}{J_a} \left[ \rho, \rho u, \rho v, \rho w, \rho e \right]^T, \text{ where } \rho e = \frac{p}{(\gamma - 1)} + \frac{1}{2}(u^2 + v^2 + w^2), \tag{5}
\]

where \( u, v \) and \( w \) are the \( x, y \) and \( z \) components of the fluid velocity. The vector \( S_{MHD} \) contains the MHD source terms,

\[
S_{MHD} = \frac{1}{J_a} \left[ 0, \ (J \times B_a)_x, \ (J \times B_a)_y, \ (J \times B_a)_z, \ E \cdot J \right]^T \tag{6}
\]

where

\[
J = \sigma_e (E + u \times B_a), \text{ with } E = -\nabla \phi, \text{ and } \nabla^2 \phi = -\nabla \cdot (u \times B_a). \tag{7}
\]

In the geometries treated here, \( u \) and \( B_a \) do not have \( \phi \) components, so \( u \times B_a \) is in the \( \phi \) direction, but since \( \partial / \partial \phi = 0 \), the electric field is actually zero. We introduce the vectors \( l, m, n \). These are the normal vectors on \( \xi, \eta, \zeta \) surfaces with their magnitudes equal to the elemental surface area and pointing to the directions of increasing \( \xi, \eta, \zeta \),

\[
l = \frac{\nabla \xi}{J_a} \Delta \eta \Delta \zeta, \quad m = \frac{\nabla \eta}{J_a} \Delta \xi \Delta \zeta, \quad n = \frac{\nabla \zeta}{J_a} \Delta \xi \Delta \eta. \tag{8}
\]

The matrices \( E, F, \) and \( G \) are

\[
E = \begin{bmatrix}
\rho u \\
\rho u + l_x p \\
\rho v u + l_y p \\
\rho w u + l_z p \\
(\rho e + p) U - l_z p 
\end{bmatrix}, \quad F = \begin{bmatrix}
\rho v \\
\rho v + m_x p \\
\rho e + m_y p \\
\rho w + m_z p \\
(\rho e + p) V - m_z p 
\end{bmatrix}, \quad G = \begin{bmatrix}
\rho w \\
\rho w + n_x p \\
\rho w + n_y p \\
\rho w + n_z p \\
(\rho e + p) W - n_z p 
\end{bmatrix}. \tag{9}
\]
In the equations above, \( U, V \) and \( W \) are the contravariant velocities in the \( \xi, \eta \) and \( \zeta \) directions,

\[
U = l_x + 1 \cdot V = l_x + l_x u + l_y v + l_z w \\
V = m_t + m \cdot V = m_t + m_x u + m_y v + m_z w \\
W = n_t + n \cdot V = n_t + n_x u + n_y v + n_z w,
\]

(10)

where \( V = (u, v, w) \) is the velocity vector. The viscosity and heat conduction flux vectors \( R, S, \) and \( T \) are

\[
R = \begin{bmatrix}
0 \\
l_k \tau_{xk} \\
l_k \tau_{yk} \\
l_k \tau_{zk} \\
l_k \beta_k
\end{bmatrix}, \quad
S = \begin{bmatrix}
0 \\
m_k \tau_{xk} \\
m_k \tau_{yk} \\
m_k \tau_{zk} \\
m_k \beta_k
\end{bmatrix}, \quad
T = \begin{bmatrix}
0 \\
n_k \tau_{xk} \\
n_k \tau_{yk} \\
n_k \tau_{zk} \\
n_k \beta_k
\end{bmatrix},
\]

(11)

where

\[
\beta_k = u_i \tau_{ki} - q_k.
\]

(12)

The shear-stress \( \tau_{ik} \) and total heat flux \( q_k \) in Cartesian Coordinate can be expressed as

\[
\tau_{ik} = \mu \left[ \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right] - \frac{2}{3} \delta_{ik} \frac{\partial u_j}{\partial x_j}, \quad \text{and}
\]

(13)

\[
q_k = - \left( \frac{\mu}{Pr} \right) \frac{\partial T}{\partial x_k}
\]

(14)

where, \( Pr \) is the Prandtl number, and \( \mu \) is the molecular viscosity determined by Sutherland’s law. In eqs.(11), (12), (13) and (14), the repeated subscripts \( i \) or \( k \) represent the coordinates \( x, y \) and \( z \) following the Einstein summation convention. Eqs.(13) and (14) are transformed to the generalized coordinate system in the computation.

V. The Numerical Scheme

The inviscid fluxes are evaluated using the LDE scheme of Ref. [8] and the fifth order WENO scheme given below. The viscous terms are discretized using a 2nd order central differencing scheme.

A. The Low Diffusion E-CUSP (LDE) Scheme

In Ref. [8] the basic idea of the E-CUSP scheme is to split the inviscid flux into the convective flux \( E^c \) and the pressure flux \( E^p \). In a generalized coordinate system, the flux \( E \) can be split as follows

\[
E = E^c + E^p = \begin{pmatrix}
\rho U \\
\rho u U \\
\rho v U \\
\rho w U \\
\rho e U
\end{pmatrix} + \begin{pmatrix}
0 \\
l_x p \\
l_y p \\
l_z p \\
p \bar{U}
\end{pmatrix}, \quad \text{where} \quad \bar{U} = l_x u + l_y v + l_z w.
\]

(15)
The convective term, $E^c$ is evaluated following the Edward’s H-CUSP (enthalpy-CUSP) LDFSS\textsuperscript{14,15}(low-diffusion-flux-splitting-scheme),

$$E^c = \rho U \begin{pmatrix} 1 \\ u \\ v \\ w \\ e \end{pmatrix} = \rho U f^c, \quad f^c = \begin{pmatrix} 1 \\ u \\ v \\ w \\ e \end{pmatrix}. \quad (16)$$

Let

$$C = c \left( l_x^2 + l_y^2 + l_z^2 \right)^{\frac{3}{2}} \quad (17)$$

where $c = \sqrt{\gamma RT}$ is the speed of sound. Then the convective flux at interface $\frac{1}{2}$ is evaluated as:

$$E^c_{\frac{1}{2}} = C_\frac{1}{2} \left[ \rho_L C^+ f^+_L + \rho_R C^- f^-_R \right], \quad (18)$$

where, the subscripts $L$ and $R$ represent the left and right hand sides of the interface. The interface speed of sound is

$$C_\frac{1}{2} = \frac{1}{2} (C_L + C_R). \quad (19)$$

The following relations are borrowed from Edwards LDFFS scheme\textsuperscript{14,15} to express the formulations from $-\infty < M < \infty$,

$$C^+ = \alpha^+ L \left( 1 + \beta_L \right) M_L - \beta_L M^+_L - M^+_L, \quad C^- = \alpha^- R \left( 1 + \beta_R \right) M_R - \beta_R M^-_R + M^+_R,$$

$$M_L = \frac{U_L}{C^+_L}, \quad M_R = \frac{U_R}{C^-_R}, \quad \alpha_{L,R} = \frac{1}{2} \left[ 1 \pm \text{sign} (M_{L,R}) \right], \quad \beta_{L,R} = - \text{max} [0, 1 - \text{int} (|M_{L,R}|)],$$

$$M^+_L = M^+_L \frac{C_{R} + C_{L}}{C_{R} + C_{L}}, \quad M^-_L = M^+_L \frac{C_{L} + C_{R} \Phi^{-1}}{C_{L} + C_{R}}, \quad \Phi = \frac{(\rho c^2)^R_R}{(\rho c^2)^L_L}, \quad M^+_R = \beta_L \delta^+ M^-_R - \beta_R \delta^- M^+_R,$$

$$M^{\pm}_{L,R} = \pm \frac{1}{2} \left( M_{L,R} \pm 1 \right)^2, \quad \text{and} \quad \delta^\pm = \frac{1}{2} \left\{ 1 \pm \text{sign} \left[ \frac{1}{2} (M_L + M_R) \right] \right\}. \quad (20)$$

The pressure flux, $E^p$ is evaluated as

$$E^p_{\frac{1}{2}} = \begin{pmatrix} 0 \\ pl_x \\ pl_y \\ pl_z \\ p\bar{U} \end{pmatrix} = \begin{pmatrix} 0 \\ (D^+_L p_L + D^+_R p_R) l_x \\ (D^+_L p_L + D^+_R p_R) l_y \\ (D^+_L p_L + D^+_R p_R) l_z \\ C_{\frac{1}{2}} (S^+_L p_L + S^-_R p_R) \end{pmatrix}, \quad (21)$$

where,

$$D^\pm_{L,R} = [\alpha (1 + \beta) - \beta P^\pm]_{L,R}. \quad (22)$$

The pressure splitting coefficient is:

$$P^\pm_{L,R} = \frac{1}{4} (M_{L,R} \pm 1)^2 (2 \mp M_{L,R}). \quad (23)$$

For the pressure term in the energy equation, the contravariant speed of sound $\bar{C}$ is consistent with $\bar{U}$ and is calculated as:

$$\bar{C} = C - U, \quad \text{with} \quad S^\pm_{L,R} = [\alpha^\pm (1 + \beta)] M - \beta M]_{L,R}, \text{ and where} \quad \bar{M} = \frac{\bar{U}}{\bar{C}}. \quad (24)$$
and $\bar{\beta}$ are evaluated based on $\bar{M}$ using the formulations given in eq. (20). The use of $\bar{U}$, $\bar{C}$, and $\bar{M}$ in the pressure term for the energy equation is to take into account the grid speed so that the flux will transit from subsonic to supersonic smoothly. When the grid is stationary, $l_i = 0$, $\bar{C} = C$, $\bar{U} = U$. The LDE scheme can accurately resolve wall boundary layer profiles, capture crisp shock profiles and exact contact surfaces with low diffusion.

B. The Weighted Essentially Non-Oscillatory (WENO) Scheme\textsuperscript{16,17}

The fifth-order accurate WENO $(r = 3)$ reconstruction of $u^L$ and $u^R$ can be written as

$$u_{i+1/2}^L = \omega_0 q_0 + \omega_1 q_1 + \omega_2 q_2$$

where

$$q_0 = \frac{1}{3} u_{i-2} - \frac{7}{6} u_{i-1} + \frac{11}{6} u_i, \quad q_1 = -\frac{1}{6} u_{i-1} + \frac{5}{6} u_i + \frac{1}{3} u_{i+1}, \quad q_2 = \frac{1}{3} u_i + \frac{5}{6} u_{i+1} - \frac{1}{6} u_{i+2}$$

and

$$\omega_s = \frac{\alpha_s}{\alpha_0 + \cdots + \alpha_{r-1}}, \quad \alpha_s = \frac{C_s}{\varepsilon + IS_s}, \quad k = 0, \cdots, r - 1, \quad C_0 = 0.1, \quad C_1 = 0.6, \quad C_2 = 0.3,$$

$$IS_0 = \frac{13}{12} (u_{i-2} - 2u_{i-1} + u_i)^2 + \frac{1}{4} (u_{i-2} - 4u_{i-1} + 3u_i)^2,$$

$$IS_1 = \frac{13}{12} (u_{i-1} - 2u_i + u_{i+1})^2 + \frac{1}{4} (u_{i+1} - u_{i-1})^2,$$

and

$$IS_2 = \frac{13}{12} (u_i - 2u_{i+1} + u_{i+2})^2 + \frac{1}{4} (3u_i - 4u_{i+1} + u_{i+2})^2$$

where, $\varepsilon = 10^{-2}$ is used following the recommendation in Shen, Zha, and Wang.\textsuperscript{16} The $u^R$ is reconstructed following the symmetry rule as that to the $u^L$ at interface $i + 1/2$. In this paper, the WENO scheme described above is used to evaluate the conservative variables with 5th order accuracy. The interface flux is then approximated with 5th order accuracy based on the approximate Riemann solver of the implicit flux.

C. The Discretization of Viscous Terms\textsuperscript{18}

A fully conservative fourth-order accurate finite central differencing scheme developed in\textsuperscript{18,19} is employed for the viscous terms. For examples

$$\left. \frac{\partial R}{\partial \xi} \right|_i = \bar{R}_{i+1/2} - \bar{R}_{i-1/2}, \quad \text{where}$$

$$\bar{R}_{i-1/2} = \frac{1}{24\Delta \xi} (-R_{i+1/2} + 26R_{i-1/2} - R_{i-3/2}), \quad R_{i-1/2} = [(\xi_x \tau_{xx}) + (\eta_y \tau_{xy}) + (\xi_z \tau_{xz})]_{i-1/2},$$

and

$$\tau_{xx} = \mu [4 \xi_x \frac{\partial u}{\partial \xi}]_{i-1/2} + (\eta_y \frac{\partial u}{\partial \eta})_{i-1/2} + (\xi_z \frac{\partial u}{\partial \xi})_{i-1/2} + (\xi_{xx} \frac{\partial u}{\partial \xi})_{i-1/2}$$

$$- \frac{2}{3} [(\xi_y \frac{\partial u}{\partial \eta})_{i-1/2} + (\eta_y \frac{\partial u}{\partial \eta})_{i-1/2} + (\xi_z \frac{\partial u}{\partial \xi})_{i-1/2} + (\xi_{xx} \frac{\partial u}{\partial \xi})_{i-1/2}]$$

$$+ (\xi_z \frac{\partial u}{\partial \xi})_{i-1/2} + (\eta_z \frac{\partial u}{\partial \eta})_{i-1/2} + (\xi_{xx} \frac{\partial u}{\partial \xi})_{i-1/2}]$$
To achieve strict 4th-order accuracy, Eqs.(26)-(27) must be used. For interface \( I = i - 3/2, i - 1/2, i + 1/2 \), the following formulas must be adopted.

\[
\mu_I = \sum_{l=m}^{n} C_I^l \mu_{i+l}, \quad m = -2, n = 1, \quad \frac{\partial u}{\partial \xi} \bigg|_I = \frac{1}{\Delta \xi} \sum_{l=r}^{s} D_I^l u_{i+l}, \quad r = -3, n = 2, \quad (28)
\]

\[
\frac{\partial u}{\partial \eta} \bigg|_I = \sum_{l=m}^{n} C_I^l \frac{\partial u}{\partial \eta} \bigg|_{i+l,j}, \quad m = -2, n = 1, \quad \text{and} \quad \frac{\partial u}{\partial \eta} \bigg|_{i,j} = \frac{1}{\Delta \eta} \sum_{l=p}^{q} C^c_I u_{i,j+l}, \quad p = -2, q = 2. \quad (29)
\]

The other terms are determined similarly. By choosing different ranges for \((m,n),(r,s),(p,q)\) and different coefficients \(C_I^l, D_I^l, C^c_I\), one can obtain the different order accuracy of the viscous terms. In this paper, we take \((m,n) = (-2,1), (r,s) = (-3,2), \text{and} \ (p,q) = (-2,2)\), then \(\mu_I, \frac{\partial u}{\partial \eta} \bigg|_I, \frac{\partial u}{\partial \eta} \bigg|_{i,j}\) achieve fourth-order accuracy, and \(\frac{\partial u}{\partial \xi} \bigg|_I\) achieve fifth-order accuracy.\(^{18}\) The coefficients \(C_I^l, D_I^l, C^c_I\) are given in Tables 1-3.

| Table 1. The coefficients of \(C_I^l\) |
|-----------------|-----------------|-----------------|-----------------|
| \(I\)           | \(C_{-2}^l\)    | \(C_{-1}^l\)    | \(C_0^l\)       | \(C_1^l\)       |
| \(i - 3/2\)     | 5/16            | 15/16           | -5/16           | 1/16            |
| \(i - 1/2\)     | -1/16           | 9/16            | 9/16            | -1/16           |
| \(i + 1/2\)     | 1/16            | -5/16           | 15/16           | 5/16            |

| Table 2. The coefficients of \(D_I^l\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(I\)           | \(D_{-3}^l\)    | \(D_{-2}^l\)    | \(D_{-1}^l\)    | \(D_0^l\)       | \(D_1^l\)       | \(D_2^l\)       |
| \(i - 3/2\)     | 71/1920         | -141/128        | 69/64           | 1/192           | -3/128          | 3/640           |
| \(i - 1/2\)     | -3/640          | 25/384          | -75/64          | 75/64           | -25/384         | 3/640           |
| \(i + 1/2\)     | -3/640          | 3/128           | -1/192          | -69/64          | 141/128         | -71/1920        |

| Table 3. The coefficients of \(C^c_I\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(C_{-2}^c\)    | \(C_{-1}^c\)    | \(C_0^c\)       | \(C_1^c\)       | \(C_2^c\)       |
| 1/12            | -8/12           | 0               | 8/12            | -1/12           |

D. Implicit Time Integration

In the current work, the finite difference method is used to discretize the governing equations for a steady state solution. To achieve high convergence rate, the implicit time marching scheme is used with the unfactored Gauss-Seidel line relaxation. To enhance diagonal dominance, the first order Euler method is used to discretize the temporal term

\[
\frac{\Delta V}{\Delta t} (Q_{n+1} - Q_n) + \left( E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}} \right)^{n+1} \left( F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} \right)^{n+1} \left( G_{k+\frac{1}{2}} - G_{k-\frac{1}{2}} \right)^{n+1} + \left( R_{i+\frac{1}{2}} - R_{i-\frac{1}{2}} \right)^{n+1} \left( S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}} \right)^{n+1} \left( T_{k+\frac{1}{2}} - T_{k-\frac{1}{2}} \right)^{n+1} + D^n \Delta V + S_m^{n+1} + S_{MHD}^{n} \Delta V. \quad (30)
\]
where $n$ and $n+1$ are two sequential time levels with a time interval of $\Delta t$. A first-order Taylor expansion for the $n+1$ time level is used for all inviscid and viscous terms above. The discretized equations are given as the following:

$$
\begin{align*}
\Delta Q_{i,j,k}^{n+1} + A^+ \Delta Q_{i+1,j,k}^{n+1} + A \Delta Q_{i,j,k}^{n+1} + A^- \Delta Q_{i-1,j,k}^{n+1} + B^+ \Delta Q_{i,j+1,k}^{n+1} + B \Delta Q_{i,j,k}^{n+1} \\
+ B^- \Delta Q_{i,j-1,k}^{n+1} + C^+ \Delta Q_{i,j,k+1}^{n+1} + C \Delta Q_{i,j,k}^{n+1} + C^- \Delta Q_{i,j,k-1}^{n+1} = RHS^n,
\end{align*}
$$

(31)

where $RHS^n$ is the summation of all the terms on the right hand side (RHS) of the equation, and

$$
RHS^n = \frac{\Delta t}{Re \Delta V} \left\{ \left[ \left( R_i^{n+\frac{1}{2}} - R_i^{n-\frac{1}{2}} \right) + \left( S_j^{n+\frac{1}{2}} - S_j^{n-\frac{1}{2}} \right) + \left( T_k^{n+\frac{1}{2}} - T_k^{n-\frac{1}{2}} \right) \right] \\
- \left[ \left( E_i^{n+\frac{1}{2}} - E_i^{n-\frac{1}{2}} \right) + \left( F_j^{n+\frac{1}{2}} - F_j^{n-\frac{1}{2}} \right) + \left( G_k^{n+\frac{1}{2}} - G_k^{n-\frac{1}{2}} \right) \right] \right\} + \frac{1}{Re} D^n \Delta t + S_{MHD}^n \Delta t.
$$

(32)

The Gauss-Seidel line relaxation is applied on each direction respectively and is swept forward and backward once within each physical time. For example, if the sweeping is in the $i$ direction from smaller index to larger one, Eq. [31] will be

$$
B^- \Delta Q_{i,j-1,k}^{n+1} + B \Delta Q_{i,j,k}^{n+1} + B^+ \Delta Q_{i,j+1,k}^{n+1} = RHS'
$$

(33)

where $B = I + A + B + C$. The terms in the neighboring cells in the $i$ and $k$ directions are absorbed into $RHS^n$ as $RHS'$,

$$
RHS' = RHS^n - A^+ \Delta Q_{i+1,j,k}^{n+1} - A^- \Delta Q_{i-1,j,k}^{n+1} - C^+ \Delta Q_{i,j,k+1}^{n+1} - C^- \Delta Q_{i,j,k-1}^{n+1}
$$

(34)

The unfactored implicit Gauss-Seidel line relaxation employed in this paper is significantly more efficient than the LU-SGS implicit scheme.  

VI. Validation

The simulations are validated by comparing the results for Mach 5 flow around a 3D hemisphere at an altitude of 40km with those in the paper by Damevin and Hoffmann. The parameters in the simulation were chosen to match those used in the above paper. The parameters are $p_\infty$=287.1Pa, $T_\infty$=250.35K, and $L_\infty$=0.01m. The flow is treated as laminar as in Damevin and Hoffmann. We compare the result for equilibrium air because in this speed regime the high temperature gas effect starts to be significant. The magnetic effects depend on the magnetic interaction parameter $Q$, where $Q = \frac{\sigma e ref B_{ref}^2 L_\infty}{\rho_\infty U_\infty}$. It is important to note that the same $Q$ is achieved with smaller $B_{ref}$ when $L_\infty$ is large. Fig. [1] shows the comparison of the pressure and the temperature distributions for interaction parameters $Q=0$ and $Q=6$. In this case $Q=6$ corresponds to $B_{ref}=2.706T$. This large $B_{ref}$ is the consequence of very small $L_\infty$. Here it can be seen that the applied magnetic field moves the shockfront away from the body increasing the shock stand-off distance, and the highest pressure and temperature regions in the vicinity of the nose of the body are significantly enlarged. Also, due to the high temperature gas effect, the stagnation temperature is lower than that of a perfect gas which would be about 6.0. We compare our results for various applied magnetic field strengths corresponding to interaction parameters from $Q = 1$ to $Q = 6$ with the results in Ref. [21]. Fig. [2] shows that the overall effect of the magnetic fields matches well that of Hoffmann’s result.
VII. Mach 12 Equilibrium Flow at Altitude of 40km

We consider Mach 12 equilibrium air flow at an altitude of 40km around a 1 m radius 3D hemisphere. We use the actual flow condition for 40km, \( \rho_\infty = 3.996 \times 10^{-3} \text{ kg/m}^3 \), \( p_\infty = 2.871 \times 10^2 \text{Pa} \), \( T_\infty = 250.35 \text{K} \), \( \mu_\infty = 1.602 \times 10^{-5} \text{Pa} \cdot \text{s} \), and \( a_\infty = 317.19 \text{m/s} \). The Reynolds number is 950,025 with \( L_\infty = 1 \text{m} \). The wall is assumed to be isothermal with wall temperature \( T_W = 1250 \text{K} \). The strength of the applied magnetic field is formulated in terms of the magnetic interaction parameter \( Q = \sigma_{e,ref} B_{ref}^2 \frac{L_\infty}{\rho_\infty U_\infty} \), with \( B_{ref} \) at the stagnation point. For the case \( Q = 16 \), \( B_{ref} = 1.56 \text{T} \). We use the ionization model of Section III with an onset temperature of 4,000K. There are 16 blocks with 128x16x16 cells in each block. Fig. [3] shows a comparison of the pressure and temperature fields for \( Q = 0 \) and \( Q = 16 \). The hottest temperature is \( T_{hot} \approx 20 \times T_\infty = 5,000 \text{K} \). Fig. [4] shows the space distribution of...
Figure 3. Comparison of pressure and temperature fields for $Q=0$ and $Q=16$.

Figure 4. $\gamma$ vs. $Q$.

$\gamma$ calculated as discussed in Section II for the two cases $Q = 0$, where there is no magnetic field, and for $Q = 16$. In front of the shock $\gamma = 1.4$, but its value drops significantly in the regions of high temperature. Fig. [5] shows the pressure and temperature profiles along the stagnation streamline. Fig. [6] shows the electrical conductivity fields for $Q = 4, 8, 12$, and $16$. With $\sigma$ given by Eq. [2], $\sigma = 0$ in front of the shock, and less than $\sigma_{e,ref}$ wherever $T < T_{ref}$, where $T_{ref}$ was defined in Section III. Fig. [7] shows the MHD force density field for $Q = 4$ to $Q = 16$. Note that the force density is zero along the stagnation streamline because there the fluid velocity and the magnetic field are antiparallel. Fig. [8] shows the variation of the shock stand-off distance from $Q = 0$ to $Q = 16$. The shock stand-off distance seems to be proportional to $Q$ up to about $Q=8$, but its proportionality changes afterwards and its overall response to $Q$ seems to be nonlinear for large $Q$. At $Q = 16$ there is an increase of almost 78% compared with the result at $Q = 0$. Fig. [9] shows the aerodynamic and MHD drag forces versus $Q$. At $Q = 16$, which has a 1.56T magnetic field at the stagnation point, the MHD drag is almost 15% of the aerodynamic drag.
Figure 5. Comparison of pressure and temperature fields along the stagnation streamline for $Q=0$ to $Q=16$.

Figure 6. Comparison of the electrical conductivity fields for $Q=4$ to $Q=16$.

Figure 7. Comparison of the MHD force density fields for $Q=4$ to $Q=16$. 
VIII. Conclusions and Future Work

We have succeeded in simulating Mach 12 turbulent 3D flow around a hemisphere using the low magnetic Reynolds number MHD model and real gas effects into the low diffusion E-CUSP (LDE) scheme with a fifth order WENO scheme recently developed by Zha\textsuperscript{8} et al. We find the same physical effects that have already been found by the authors already cited. We are planning to have results with nonequilibrium air, and a realistic electrical conductivity in future work.
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